Darren Forde

(SLAC & UCLA)

ON-SHELL METHODS FOR ONE-LOOP AMPLITUDE CALCULATIONS

arXiv:0704.1835 [hep-ph], hep-ph/0607014, hep-ph/0604195

In collaboration with Carola Berger, Zvi Bern, Lance Dixon & David Kosower.

OVERVIEW

Why do we need one-loop amplitudes?

- Limitations of standard techniques.

The "Unitarity bootstrap technique" – an efficient method for calculating one-loop amplitudes.

Construct amplitude in two pieces,

- On-shell recursion relations,
- Generalised unitarity techniques.

Focus on generalised unitarity techniques/direct extraction methods.

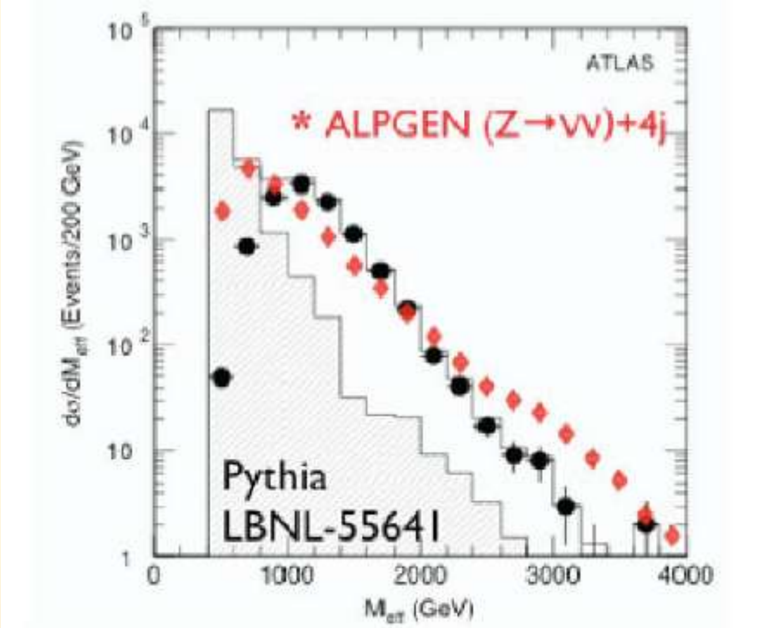
WHAT'S THE PROBLEM?

- QCD amplitudes are needed to understand the results from colliders– Tevatron and LHC (2007).

PRECISE QCD CALCULATIONS

- Probe beyond the Standard Model,
  - New particles typically decay into old particles,
  - Signals in discovery channels can be close to backgrounds,
  - Maximise discovery potential

    $\Rightarrow$ Precise understating of background processes.
- Measurements of
  - fundamental parameters (α_s, m_t),
  - Luminosity,
  - Extraction of parton distributions, etc.



SUSY search:
Early ATLAS TDR
using PYTHIA.

WHAT DO WE NEED?

"Famous" Les Houches list, (2005)

process ($V \in \{Z, W, \gamma\}$)	background to
1. $pp \to V\,V$ jet	$t\bar{t}H$, new physics
2. $pp \to H + 2$ jets	H production by vector boson fusion (VBF)
3. $pp \to t\bar{t}\,b\bar{b}$	$t\bar{t}H$
4. $pp \to t\bar{t} + 2$ jets	$t\bar{t}H$
5. $pp \to V\,V\,b\bar{b}$	VBF$\to H \to VV$, $t\bar{t}H$, new physics
6. $pp \to V\,V + 2$ jets	VBF$\to H \to VV$
7. $pp \to V + 3$ jets	various new physics signatures
8. $pp \to V\,V\,V$	SUSY trilepton

WW+j: Campbell, Ellis, Zanderighi.
Dittmaier, Kallweit, Uwer.

VBF: Bozzi, Jager, Oleari, Zeppenfeld.

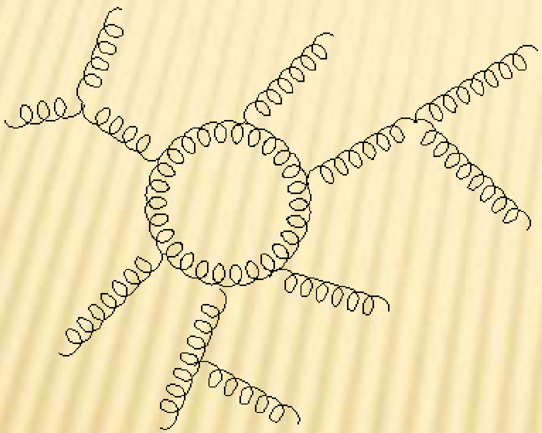
ZZZ: Lazopoulos, Petriello, Melnikov

Five, six or more legs.

WHAT'S THE HOLD UP?

Gauge dependant quantities,
large cancellations between terms.

- Calculating using Feynman diagrams is Hard!
- A Factorial growth in the number of terms.

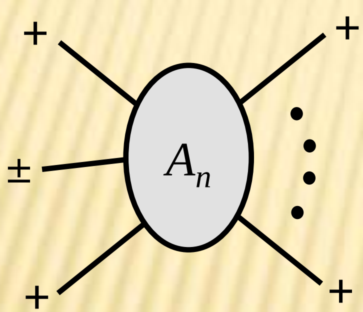


6 gluons	~10,000 diagrams.
7 gluons	~150,000 diagrams.
8 gluons	more than **3 million.**
n gluons	∞ diagrams.

- Numerical approaches [von Hameren, Vollinga, Weinzierl], [Giele, Glover], [Giele, Glover, Zanderighi], [Ellis, Giele, Zanderighi], [Binoth, Guillet, Heinrich, Pilon, Schubert]
- More efficient techniques desired.

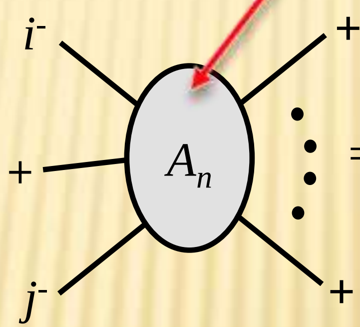
SIMPLE RESULTS!

- Calculated amplitudes simpler than expected.
- For example, tree level all-multiplicity gluon amplitudes [Parke, Taylor]



$=0$

MHV Amplitude



Spinor helicity notation

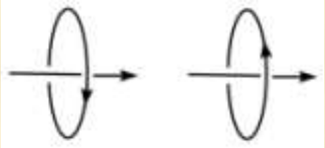
$$= \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle (n-1)n\rangle\langle n1\rangle}$$

- The problem with Feynman diagrams
  - Gauge dependent,
  - Contain off-shell vertices and propagators.
- Want to use on-shell quantities only.

SPINOR HELICITY METHOD

- Appropriate choice of variables gives simpler/more compact results.
- Write amplitude using spinors, objects with definite helicity $h=\pm 1$.

$$\lambda_i \equiv \left| i^+ \right\rangle \equiv u_+(k_i), \qquad \tilde{\lambda}_i \equiv \left| i^- \right\rangle \equiv u_-(k_i)$$



- Rewrite all vectors in terms of spinors e.g. polarisation vectors [Xu, Zhang, Chang]

$$\varepsilon_\mu^+(k,q) = \frac{\left\langle q^- \middle| \gamma_\mu \middle| k^- \right\rangle}{\sqrt{2}\left\langle q^- \middle| k^+ \right\rangle} \text{ and } \varepsilon_\mu^-(k,q) = \frac{\left\langle q^+ \middle| \gamma_\mu \middle| k^+ \right\rangle}{\sqrt{2}\left\langle q^+ \middle| k^- \right\rangle}.$$

- Amplitude is now written entirely in terms of spinors.

SPINOR PRODUCTS

- Two different spinor products,

$$\bar{u}_-(k_1)u_+(k_2) \equiv \left\langle 1^- \middle| 2^+ \right\rangle \equiv \langle 12 \rangle \equiv \varepsilon_{ab}\lambda_1^a \lambda_2^b$$

$$\bar{u}_+(k_1)u_-(k_2) \equiv \left\langle 1^+ \middle| 2^- \right\rangle \equiv [12] \equiv \varepsilon_{ab}\tilde{\lambda}_1^a \tilde{\lambda}_2^b$$

- Spinors related to 4-vectors

$$k^\mu = \sigma_{ab}^\mu \lambda_k^a \tilde{\lambda}_k^b \text{ and } \not{k} = u(k)\bar{u}(k).$$

- Spinor products related to Lorentz products

$$\langle ab \rangle [ba] = s_{ab} = (k_a + k_b)^2$$

COMPLEX SPINORS

- For complex spinors $\bar{\lambda}_p \neq \pm\tilde{\lambda}_p$.
  - Spinor products are independent $\langle ab\rangle \not\propto [ba]$.
- Some 3-point vertices no longer vanish,
  - Momentum conservation $\Rightarrow p_1.p_2 = p_2.p_3 = p_1.p_3 = 0$.
  - For real momentum

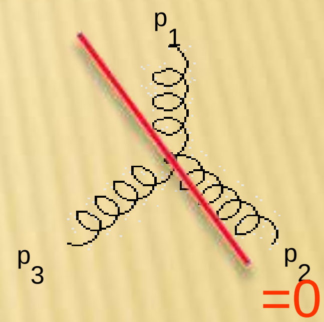
$$p.q = \langle pq\rangle[qp] = 0 \;\; \Rightarrow \;\; \langle pq\rangle = 0 \text{ and } [qp] = 0.$$

  - For complex momentum

$$p.q = \langle pq\rangle[qp] = 0 \;\; \Rightarrow \;\; \langle pq\rangle = 0 \text{ or } [qp] = 0.$$

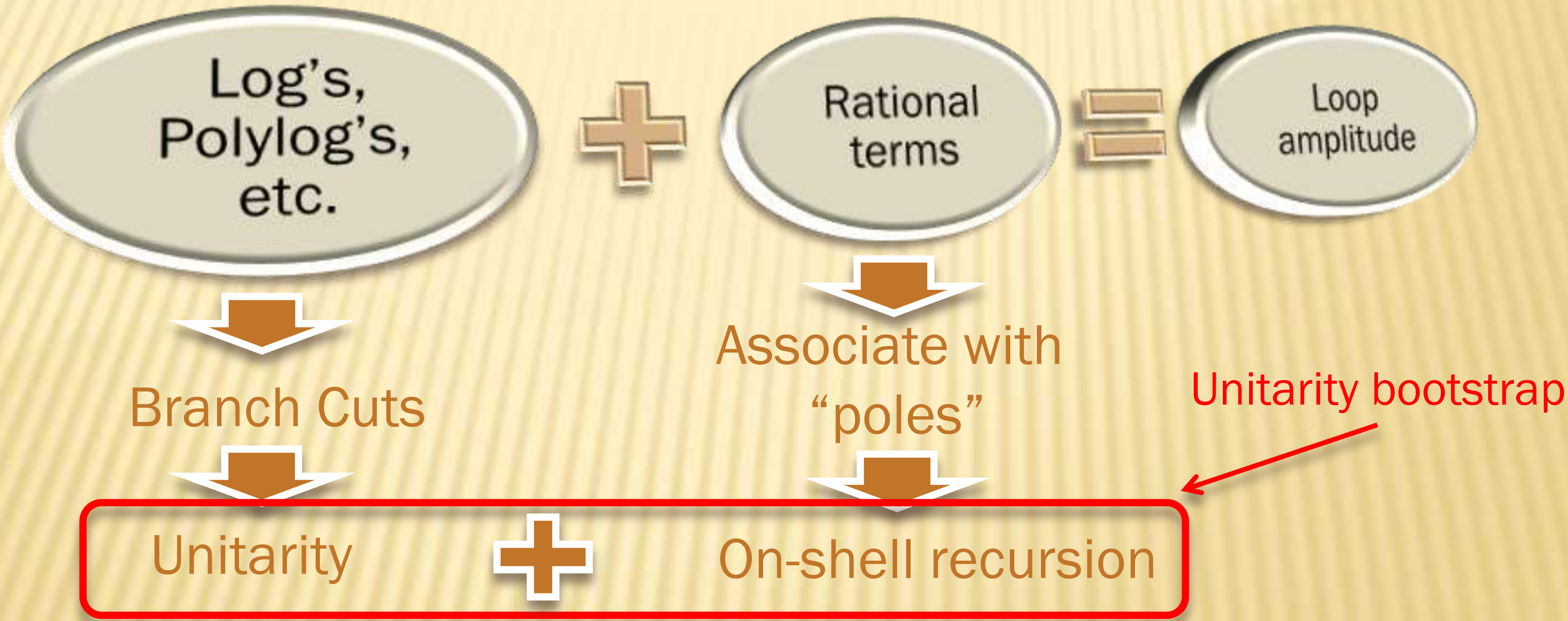
- The 3-point vertex can survive, e.g. for gluons

$$A_3(p^-, q^-, r^+) = \frac{i\langle pq\rangle^3}{\langle qr\rangle\langle rp\rangle} = 0 \;\text{ or }\; A_3(p^+, q^+, r^-) = \frac{i[pq]^3}{[qr][rp]} = 0.$$



STRUCTURE OF A 1-LOOP AMPLITUDE

- The analytic form of a 1-loop amplitude is made up of



- Consider the amplitude as a "function" on the complex plane, it will contain branch cuts and poles.
- Use the most appropriate technique for each piece.

AMPLITUDES AND THE COMPLEX PLANE

- An amplitude is a function of its external momenta (and helicity)

$$A_n(k_1^{h_1}, k_2^{h_2}, \ldots, k_i^{h_i}, \ldots, k_j^{h_j}, \ldots, k_n^{h_n})$$

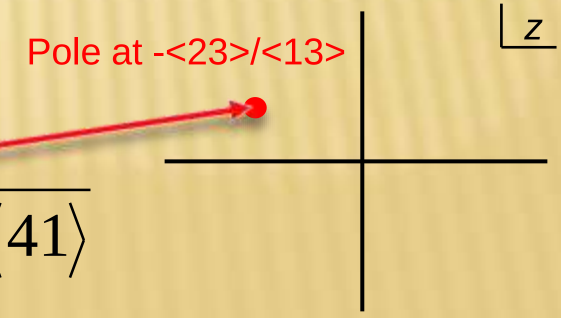
$$\hat{i}^\mu(z) = i^\mu - \frac{z}{2}\langle i^- | \gamma^\mu | j^- \rangle, \quad \hat{j}^\mu(z) = j^\mu + \frac{z}{2}\langle i^- | \gamma^\mu | j^- \rangle$$

- Shift the momentum of two external legs by a complex variable z,
  [Britto, Cachazo, Feng, Witten]
  - Keeps both k_i and k_j on-shell.
  - Conserves momentum in the amplitude.

Only possible with Complex momenta.

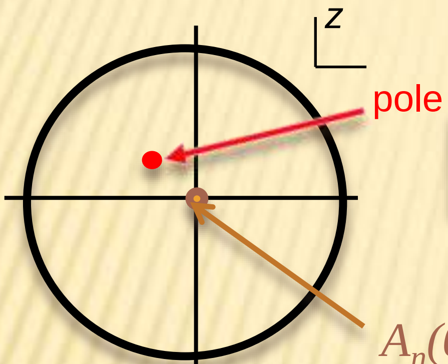
- For example

$$A_4^{\text{tree}}(\hat{1}^-, \hat{2}^+, 3^-, 4^+) = \frac{\langle \hat{1}3\rangle^4}{\langle \hat{1}\hat{2}\rangle\langle \hat{2}3\rangle\langle 34\rangle\langle 4\hat{1}\rangle}$$



A SIMPLE IDEA

- Function of a complex variable containing only simple poles



$$\frac{1}{2i\pi}\oint_C dz \frac{A_n(z)}{z} = 0 \quad \Rightarrow \quad A_n(0) = -\sum_{\text{poles}} \text{Res}_z \frac{A_n(z)}{z}$$

- Position of all poles from complex factorisation properties of the amplitude.

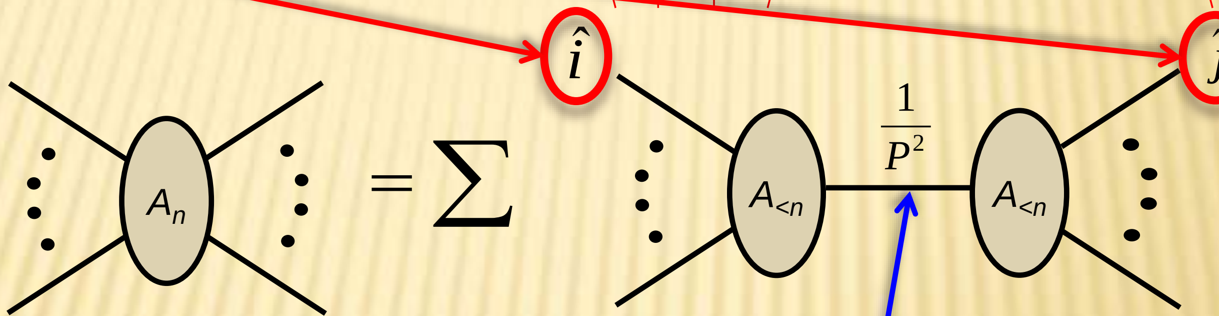
Poles from here

$$A_n \xrightarrow{P^2 \to 0} \sum_{i\in L,\, j\in R} A_L(\ldots,\hat{i}(z),\hat{P}(z)) \frac{1}{P^2} A_R(\ldots,\hat{j}(z),\hat{P}(z))\Bigg|_{z=\frac{P^2}{\langle i^-|P|j^-\rangle}} = \sum_{\text{poles}} \text{Res} \frac{A_n(z)}{z}$$

ON-SHELL RECURSION RELATIONS

- Recursion using on-shell amplitudes with fewer legs,

Two reference legs "shifted", $i^\mu \to i^\mu - \frac{P^2}{\langle i^- | \not{P} | j^- \rangle}\langle i^- | \gamma^\mu | j^- \rangle,\ j^\mu \to j^\mu + \frac{P^2}{\langle i^- | \not{P} | j^- \rangle}\langle i^- | \gamma^\mu | j^- \rangle$



Intermediate momentum leg is on-shell.

- Final result independent of choice of shift.
- Complete amplitude at tree level. [Britto, Cachazo, Feng] +[Witten]

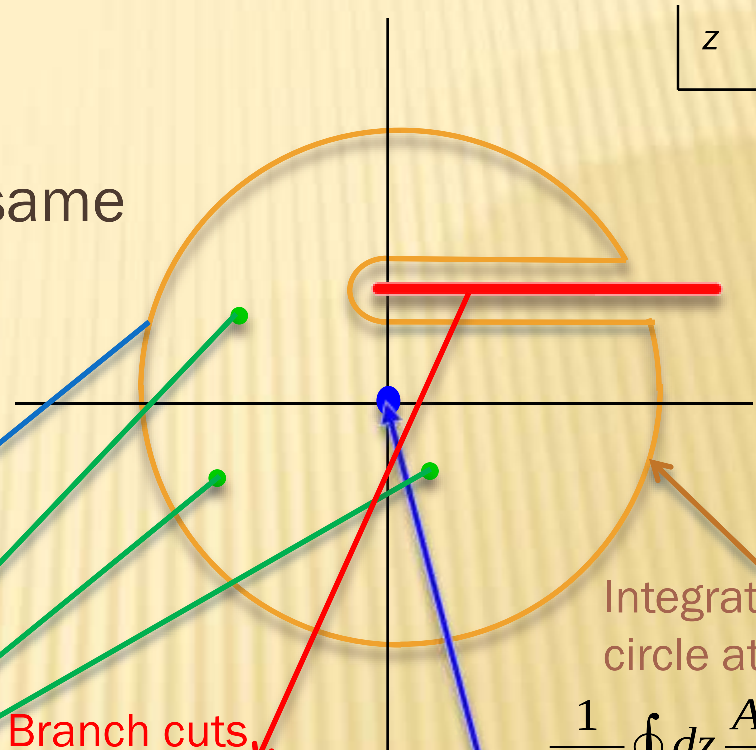
BRANCH CUTS

- What about loops?
- Shift the amplitude in the same way

$$A_n^1(k_1^{h_1},\ldots,k_i^{h_i}(z),\ldots,k_j^{h_j}(z),\ldots,k_n^{h_n})$$

$$\hat{j}^\mu = j^\mu + \frac{z}{2}\langle i^-|\gamma^\mu|j^-\rangle$$

$$\hat{i}^\mu = i^\mu - \frac{z}{2}\langle i^-|\gamma^\mu|j^-\rangle$$



Integrate over a circle at infinity

$$\frac{1}{2i\pi}\oint_C dz\,\frac{A_n(z)}{z} = 0$$

Contribution from circle at infinity

Poles

Branch cuts

$$[\mathrm{Inf}\,A_n](z) - \sum_{\text{poles}} \mathrm{Res}_z \frac{A_n(z)}{z} + \int_B^\infty \frac{dz}{z}\mathrm{Disc}_B A(z) = A(0)$$

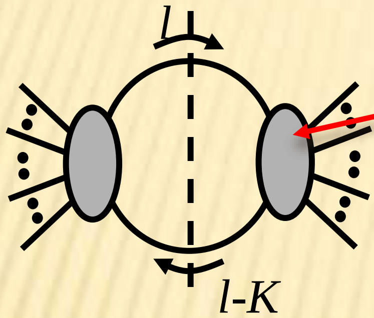
From on-shell recurrence relation

From on-shell recurrence relation

From Unitarity techniques

UNITARITY CUTTING TECHNIQUES

- Basic idea, glue together tree amplitudes to form loops. [Bern,Dixon,Dunbar,Kosower]



On-shell tree amplitudes
$\Rightarrow$ Compact loop result

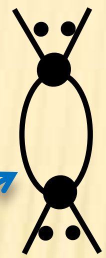
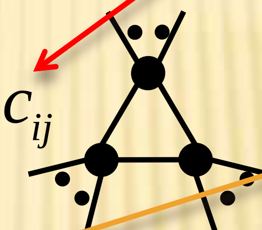
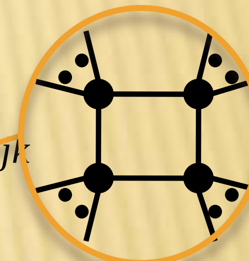
- "Cut-constructible" terms from gluing together trees in $D=4$, [Bern, Dixon, Dunbar, Kosower]
  - Missing rational pieces in QCD $\Rightarrow$ use on-shell recursion.
- Alternatively work in $D=4\text{-}2\varepsilon$, [Bern, Morgan], [Anastasiou, Britto, Feng, Kunszt, Mastrolia]
  - Gives both terms but requires trees in $D=4\text{-}2\varepsilon$.
- Extract "cut-constructible" pieces in the most efficient way.

ONE-LOOP INTEGRAL BASIS

- A one-loop amplitude decomposes into

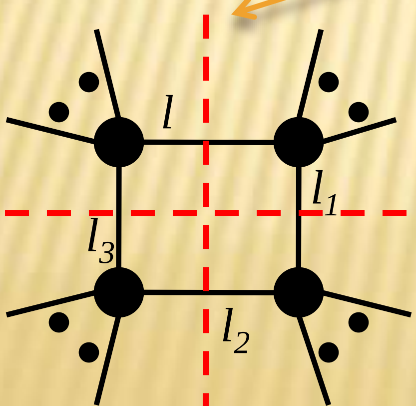
Rational terms

Rational coefficients

$$R_n + r_\Gamma \frac{(\mu^2)^\varepsilon}{(4\pi)^{2-\varepsilon}} \left(\sum_i b_i \right.$$  $$+ \sum_{ij} c_{ij}$$  $$\left. + \sum_{ijk} d_{ijk} \right.$$  $$\Bigg)$$

1-loop scalar integrals

- Quadruple cuts freeze the integral $\Rightarrow$ boxes [Britto, Cachazo, Feng]

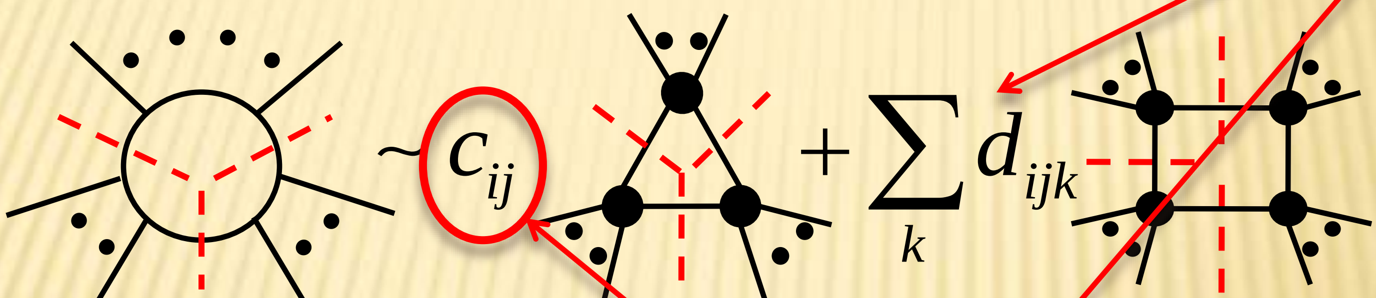


$$l^2 = l_1^2 = l_2^2 = l_3^2 = 0 \Rightarrow l_{ijk}$$

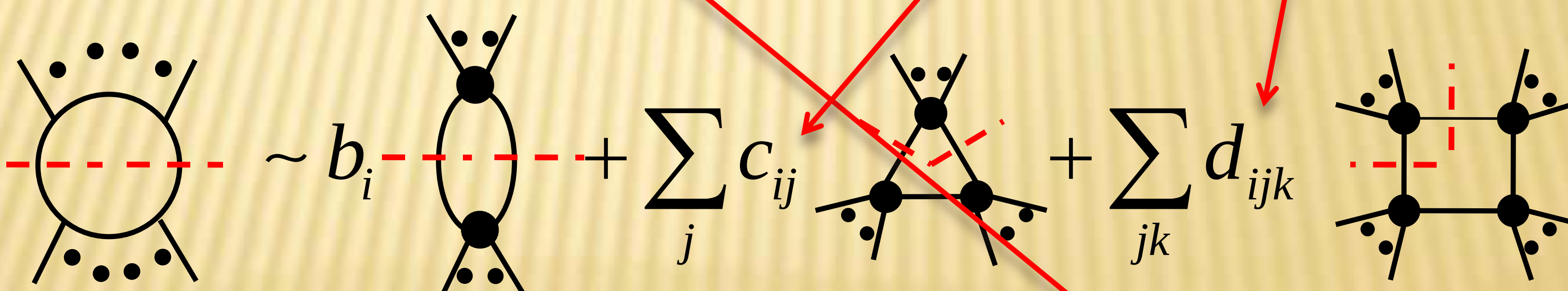
$$d_{ijk} = \frac{1}{2}\sum_{a=1}^{2} A_1(l_{ijk;a})\, A_2(l_{ijk;a})\, A_3(l_{ijk;a})\, A_4(l_{ijk;a})$$

TWO-PARTICLE AND TRIPLE CUTS

- What about bubble and triangle terms?
- Triple cut $\Rightarrow$ Scalar triangle coefficients?



- Two-particle cut $\Rightarrow$ Scalar bubble coefficients?



- Disentangle these coefficients.

Isolates a single triangle

DISENTANGELING COEFFICIENTS

- Approaches,
  - Unitarity technique, [Bern, Dixon, Dunbar, Kosower]
  - MHV vertex techniques, [Bedford, Brandhuber, Spence, Traviglini], [Quigley, Rozali]
  - Unitarity cuts & integration of spinors, [Britto, Cachazo, Feng] + [Mastrolia] + [Anastasiou, Kunszt]
  - Recursion relations, [Bern, Bjerrum-Bohr, Dunbar, Ita]
  - Solving for coefficients, [Ossola, Papadopoulos, Pittau], [Ellis, Giele, Kunszt]
- Large numbers of processes required for the LHC,
  - Automatable and efficient techniques desirable.

TRIANGLE COEFFICIENTS

- Coefficients, c_{ij}, of the triangle integral, $C_0(K_i,K_j)$, given by

$$c_{ij} = -\left[\mathrm{Inf}\, A_1 A_2 A_3\right](t)\Big|_{t\to 0}$$

Masslessly Projected momentum

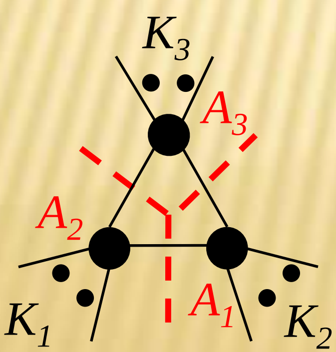
$$K_1^\flat = K_1 - \frac{S_1}{\gamma} K_2^\flat, \quad K_2^\flat = K_2 - \frac{S_2}{\gamma} K_1^\flat$$

Single free integral parameter in l

$$l^\mu = \frac{S_2(\gamma - S_1)}{\gamma^2 - S_1 S_2} K_1^{\flat\mu} + \frac{S_1(\gamma - S_2)}{\gamma^2 - S_1 S_2} K_2^{\flat\mu} + \frac{t}{2}\left\langle K_1^{\flat-}\right|\gamma^\mu\left|K_2^{\flat-}\right\rangle + \frac{S_1 S_2(\gamma - S_1)(\gamma - S_2)}{2t\left(\gamma^2 - S_1 S_2\right)^2}\left\langle K_1^{\flat+}\right|\gamma^\mu\left|K_2^{\flat+}\right\rangle$$

Triple cut of the triangle $C_0(K_i,K_j)$

$$\gamma = \left\langle K_1^{\flat-}\right|\not{K}_2^\flat\left|K_1^{\flat-}\right\rangle$$



Series expansion around t at infinity, take only non-negative powers $\Rightarrow$

$$a_0 + a_1 t^1 + a_2 t^2 + \ldots + a_{\max} t^{\max}$$

=3 in renormalisable theories

[Nagy, Soper], [Binoth, Heinrich, Gehrmann, Mastrolia], [Ossola, Papadopoulos, Pittau]

SIX PHOTONS

6 λ's top and bottom

- 3-mass triangle of $A_6(-+-+-+) \Rightarrow$ the triple cut integrand

$$16A(-l^{-h},1^-,2^+,l_2^{h_2})A(-l_2^{-h_2},3^-,4^+,l_1^{h_1})A(-l_1^{-h_1},5^-,6^+,l^h)$$

$$\frac{\langle l1\rangle^2\langle l_2 3\rangle^2\langle l_1 5\rangle^2}{\langle l2\rangle\langle l_2 2\rangle\langle l_1 4\rangle\langle l_2 4\rangle\langle l6\rangle\langle l_1 6\rangle}$$

Extra propagator $\Rightarrow$ Box terms

No propagator $\Rightarrow$ Triangle

Propagator $\leftrightarrow$ pole in t

$$\langle l2\rangle = t\langle K_1^\flat 2\rangle + \frac{S_1(\gamma-S_2)}{\gamma^2-S_1S_2}\langle K_2^\flat 2\rangle$$

$\Rightarrow$ a box.

$$\frac{32}{2}\sum_{\gamma_\pm}\int dt\,\frac{\langle K_1^\flat 1\rangle^2\langle K_1^\flat 3\rangle^2\langle K_1^\flat 5\rangle^2}{\langle K_1^\flat 2\rangle\langle K_1^\flat 4\rangle^2\langle K_1^\flat 6\rangle^2}$$

$$\int dt\left(a_{-\max}t^{-\max}+\ldots+a_{-1}t^{-1}+a_0+a_1t+\ldots+a_{\max}t^{\max}\right)+\sum_{\text{boxes}\,\{i\}}\sum_{\text{poles}\,\{j\}}\frac{\text{Res}_{t=t_j}A_1A_2A_3}{t-t_j}$$

2 solutions to $\gamma \Rightarrow$ divide by 2

The scalar triangle coefficient

- The complete coefficient.

VANISHING INTEGRALS

From series expanding the box poles

- In general series expansion of $A_1A_2A_3$ around $t=\infty$ gives,

$$\sum_{i=-\infty}^{-1} \cancel{a_i t^i} + a_0 \int dt + a_1 \int dt\cancel{t} + \ldots + a_{\max} \int dt\cancel{t^{\max}}$$

- Integrals over t vanish for chosen parameterisation, e.g.
  (Similar argument to [Ossola, Papadopoulos, Pittau])

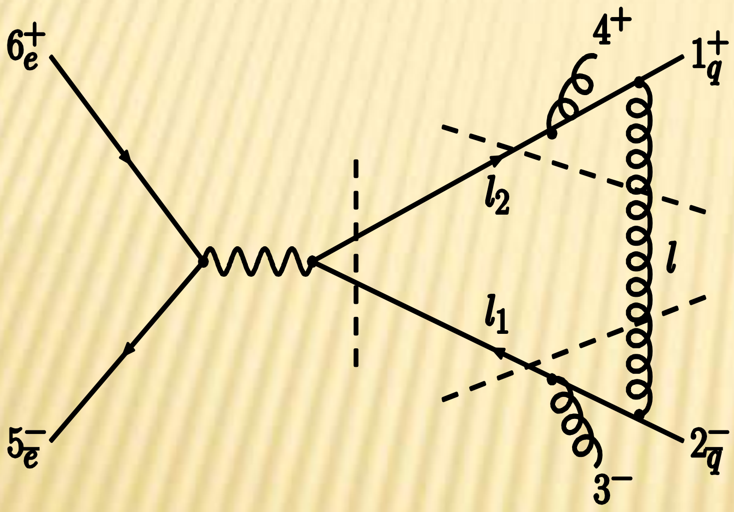
$$\int dt\,t \sim \int d^4 l \frac{\langle K_1^{\flat -} | \not{l} | K_2^{\flat -} \rangle}{l^2 l_1^2 l_2^2} \sim \langle K_1^{\flat -} | \cancel{K_1} | K_2^{\flat -} \rangle C_1 + \langle K_1^{\flat -} | \cancel{K_2} | K_2^{\flat -} \rangle C_2 = 0$$

- In general whole coefficient given by

$$c_{ij} = -\left[\mathrm{Inf}\ A_1 A_2 A_3\right](t)\Big|_{t\to 0}$$

ANOTHER TRIANGLE COEFFICIENT

- 3-mass triangle coefficient of $e^{+}\overline{e}^{-} \rightarrow q^{+}\overline{q}^{-}g^{-}g^{+}$ in the 14:23:56 channel. [Bern, Dixon, Kosower]



t Dependence

$$-i\frac{\langle l_2 5\rangle^2\langle l l_2\rangle^2\langle 23\rangle^2}{\langle 14\rangle\langle 56\rangle\langle 4 l_2\rangle\langle 2 l\rangle\langle l l_1\rangle\langle l_1 l_2\rangle}$$

Independent of t

Series expand in t around infinity

$$-\frac{i}{2}\sum_{\gamma_\pm}\frac{\gamma\langle K_1^\flat 5\rangle^2\langle 23\rangle^2}{S_1\left(1-\frac{S_1}{\gamma}\right)\langle 14\rangle\langle 56\rangle\langle 4K_1^\flat\rangle\langle 2K_1^\flat\rangle}$$

WHAT ABOUT BUBBLES?

- Can we do something similar?
- Two delta function constraints $\Rightarrow$ two free parameters y and t,

$$l^{\mu} = yK_1^{\flat\mu} + \frac{S_1(1-y)}{\gamma}\chi^{\mu} + \frac{t}{2}\left\langle K_1^{\flat-}\middle|\gamma^{\mu}\middle|\chi^{\mu}\right\rangle + \frac{S_1 y(1-y)}{2\gamma t}\left\langle K_1^{\flat+}\middle|\gamma^{\mu}\middle|\chi^{+}\right\rangle$$

- Depends upon an arbitrary massless four vector χ.
- Naive generalisation, two particle cut $\Rightarrow$ bubble coefficient b_j of the scalar bubble integral $B_0(K_j)$?

$$b_j \neq \left[\mathrm{Inf}\left[\mathrm{Inf}\; A_1 A_2\right](y)\right](t)$$

- Does not give the complete result.

VANISHING INTEGRALS?

- Series expanding around ∞ in y and then t gives

$$\cancel{\sum_{i=-\infty}^{0}}\sum_{j=-\infty}^{-1} a_{ij} t^i y^j + a_{00} + a_{10} y^1 + a_{20} y^2 + \dots + a_{\max y 0} y^{\max y}$$

$$\sum_{j=-\infty}^{-1} a_{0j} y^j + a_{00} + \frac{a_{10}}{2} + \frac{a_{20}}{3} + \dots + \frac{a_{\max y 0}}{\max y_0 + 1}$$

$$\cancel{+t\left(a_{01} + a_{11} y^1 + a_{21} y^2 + \dots + a_{\max 1} y^{\max y}\right)}$$

$$\cancel{+ \dots + t^{\max t}\left(a_{0\max t} + a_{1\max t} y^1 + a_{2\max t} y^2 + \dots + a_{\max y \max t} y^{\max y}\right)}$$

$$-i^1\left[\mathrm{Inf}\left[\mathrm{Inf}\, y^2 A_1 A_2\right](y)\, q(t)\right]\Big|_{t\to 0,\, y^m \to \frac{1}{m+1}}$$

Additional bubble contributions?

Do the poles correspond to only triangles/boxes?

- Integrals over t vanish

$$\int dt\, t^i = 0 \text{ and } \int dt \frac{1}{t^i} = 0$$

- Integrals over y do not vanish, can show

$$\int dy\, y^i = \frac{1}{i+1}\int dy$$

MISSING CONTRIBUTIONS

- Integrals over t can be related to bubble contributions.
- Schematically we write the two-particle cut integrand as,

Terms with poles in y

$$a_0(t)+a_1(t)y+\cdots+a_{\max y}(t)y^{\max y}+\sum_i \frac{\mathrm{Res}_{y=y_i} A_L\left(l(y,t)\right)A_R\left(l(y,t)\right)}{y-y_i}$$

in the residue terms $l(y_i,t)$

~"Inf" terms

y fixed at pole y_i

$$l^\mu = y_i K_1^{\flat\mu} + \frac{S_1(1-y_i)}{\gamma}\chi^\mu + \frac{t}{2}\left\langle K_1^{\flat-}\middle|\gamma^\mu\middle|\chi^-\right\rangle + \frac{S_1 y_i(1-y_i)}{2\gamma t}\left\langle K_1^{\flat+}\middle|\gamma^\mu\middle|\chi^+\right\rangle$$

- Want to associate pole terms with triangles (and boxes).
- Unlike for previous triangle coefficients though,

$$\int dt \sim \left\langle K_1^{\flat-}\middle|\not{K}_1\middle|\chi^-\right\rangle C_1 + \left\langle K_1^{\flat-}\middle|\not{K}_2\middle|\chi^-\right\rangle C_2 \neq 0$$

Integrals over t do not vanish in this expansion

$\Rightarrow$ can contain bubbles

AN EXAMPLE

- Extract the bubble coeff in three-mass linear triangle,

$$\int d^4 l \, \frac{\langle a^- | \not{l} | b^- \rangle}{l^2 (l-K_1)^2 (l-K_2)^2}$$

- Cut l^2 and $(l\text{-}K_1)^2$ propagators, gives integrand

$$\frac{\langle a^- | \not{l} | b^- \rangle}{(l-K_2)^2}$$

Series expand y and then t around ∞, set $t \to 0, \;\; \mathrm{y}^{\mathrm{m}} \to \frac{1}{m+1}$

$$-i \, \frac{\langle a \chi \rangle \left[K_1^{\flat} b \right]}{\langle \chi^- | \not{K}_2 | K_1^{\flat -} \rangle}$$

- Depends upon χ and is not the complete coefficient.

REMAINING PIECES

- Consider all triangles sitting "above" the bubble.
- Then extract bubble term from the integrals over *t*,
  - i.e. using

$$-\frac{1}{2}\sum_{\{C_{\mathrm{tri}}\}}\left[\mathrm{Inf}_t A_1 A_2 A_3\right](t)\Big|_{t^i \to T(i)}$$

  - Integrals over *t* known, (C_{ij} a constant, e.g. $C_{11}=1/2$)

$$\int dt\, t^i = \int dt\, T(i) = \left(\frac{S_1}{\gamma}\right)^i \frac{\left\langle \chi^- \middle| \not{K}_2 \middle| \hat{K}_1^- \right\rangle^i (K_1.K_2)^{i-1}}{(K_1.K_2)^2 - S_1 S_2} \sum_{j=1}^{i} C_{ij} \frac{S_2^{\,j-1}}{(K_1.K_2)^{j-1}} B_0(K_1^2)$$

  - Renormalisable theories, max power t^3.
- Combining both "two-particle" cut terms and "triple-cut" terms gives the coefficient.

FINISHING OFF THE EXAMPLE

- Setting $\chi = a$ puts the two-particle cut contribution to zero

$$-i\frac{\langle a\chi\rangle\left[K_1^\flat b\right]}{\left\langle\chi^-\middle|\not{K}_2\middle|K_1^{\flat-}\right\rangle}\overset{\chi\to a}{\Rightarrow}0$$

Only a single power of t

- We have one term from the “triple-cut” pieces,

$$it\langle a^-|\not{K}_1|b^-\rangle\left(\frac{\gamma\langle K_1^{\flat,-}|\not{K}_2|K_1^{\flat,-}\rangle-S_1\langle a^-|\not{K}_2|a^-\rangle}{S_1\langle a^-|\not{K}_2|K_1^{\flat,-}\rangle}+2\frac{[ab]}{[K_1^\flat b]}\right)$$

- The integral over t is related to the bubble via,

$$t\to\left(\frac{S_1}{2\gamma}\right)\frac{\left\langle\chi^-\middle|\not{K}_2\middle|\hat{K}_1^-\right\rangle}{(K_1.K_2)^2-S_1S_2}B_0(K_1^2)$$

- So the complete coefficient is given by

$$\frac{1}{2}\left(\frac{(K_1.K_2)\left\langle a^-\middle|\not{K}_1\middle|b^-\right\rangle}{(K_1.K_2)^2-S_1S_2}-\frac{S_1\left\langle a^-\middle|\not{K}_2\middle|b^-\right\rangle}{(K_1.K_2)^2-S_1S_2}\right)$$

A QUICK RECAP

- Calculate one-loop integral coefficients,
  - For Triangles

$$c_{ij} = -\left[\mathrm{Inf}\ A_1 A_2 A_3\right](t)\Big|_{t\to 0}$$

  - For Bubbles

$$b_i = -i\left[\mathrm{Inf}\left[\mathrm{Inf}\ A_1 A_2\right](y)\right](t)\Big|_{t\to 0,\, y^m \to \frac{1}{m+1}} - \frac{1}{2}\sum_{\{C_{\mathrm{tri}}\}}\left[\mathrm{Inf}_t A_1 A_2 A_3\right](t)\Big|_{t^i \to T(i)}$$

  - Boxes from quadruple cuts (four cuts freeze the integral).

FURTHER EXAMPLES

- Comparisons against the literature
  - Two minus all gluon bubble coefficients for up to 7 legs. [Bern, Dixon, Dunbar, Kosower], [Bedford, Brandhuber, Spence, Travigini]
  - $N=1$ SUSY gluonic three-mass triangles for $A_6(+-+-+-)$, $A_6(+-++--)$. [Britto, Cachazo, Feng]
  - Various bubble and triangle coefficients for processes of the type $e^+\overline{e}^- \to q^+\overline{q}^- g^- g^+$. [Bern, Dixon, Kosower]
- Analyses of the behaviour of one-loop gravity amplitudes, including $N=8$ Supergravity. [Bern, Carrasco, DF, Ita, Johansson]

"ON-SHELL BOOTSTRAP APPROACH"

- Has been used to calculate
  - 2 minus all-multiplicity amplitudes [DF, Kosower] [Berger, Bern, Dixon, DF, Kosower]

  - 3-minus split helicity amplitude. [Berger, Bern, Dixon, DF, Kosower]
- Important contributions to the complete analytic form for the 6 gluon amplitude [Bern,Dixon,Kosower] [Berger,Bern,Dixon,DF,Kosower] [Xiao,Yang,Zhu] [Bedford,Brandhuber,Spence,Travaglini] [Britto,Feng,Mastrolia] [Bern,Bjerrum-Bohr,Dunbar,Ita], (Numerical result [Ellis, Giele, Zanderighi])
- Small growth in complexity of solutions as number of external legs increases.

CONCLUSION

Unitarity bootstrap approach

combines

- On-shell recursion relations.
- Unitarity techniques.

Direct extraction of coefficients in 2 simple steps,

- Specific momentum parameterisation.
- Series expansion in free parameters at infinity.

Leads to an ***automatable*** procedure for one-loop computation.

"One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane." In J. Schwinger, "Particles, Sources and Fields", Vol. I.